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1982 J. Phys. A: Math. Gen. 15 2017

(http://iopscience.iop.org/0305-4470/15/7/014)

J. Phys. A: Math. Gen. 15 (1982) 2017–2040. Printed in Great Britain

On the structure of nonlinear evolution equations integrable by the Z_2-graded quadratic bundle

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Received 4 January 1982

Abstract. The general form of nonlinear evolution equations and their Bäcklund transformations connected with the quadratic in the spectral parameter, Z_2-graded, arbitrary-order linear matrix spectral problem is found. The Hamiltonian structure of the integrable equations is discussed. The infinite family of Poisson brackets which corresponds to the class of equations under consideration is given. Relativistic-invariant integrable equations are considered. The explicit forms of elementary and soliton Bäcklund transformations are found. A nonlinear superposition principle is obtained.

1. Introduction

The inverse scattering transform (IST) method permits us to investigate a large number of various partial differential equations (see e.g. Zakharov *et al* 1980, Bullough and Caudrey 1980). One of the main problems of the IST method is the problem of description of the class of equations to which this method is applicable. A very simple and convenient description of the equations integrable by the second-order linear bundle was given by Ablowitz *et al* (AKNS) (1974). The method proposed by AKNS (AKNS method) has been generalised to the linear bundle of arbitrary matrix order (Miodek 1978, Newell 1979, Kulish 1980a, Konopelchenko 1980a, b, 1981a, c), the second-order quadratic bundle (Gerdjikov *et al* 1980) and an arbitrary polynomial bundle (Konopelchenko 1981d). The linear bundle with Z_2 grading has been also considered by Konopelchenko (1980c). In the framework of the AKNS method one can construct the infinite-dimensional groups of Bäcklund transformations and investigate the Hamiltonian structure of the whole classes of the integrable equations. These are important advantages of the AKNS method.

In the present paper we consider the quadratic bundle

$$\frac{1}{i}\frac{\partial\psi}{\partial x}=(\alpha\lambda^2+2\beta\lambda)A\psi+(\alpha\lambda+\beta)P(x,t)\psi \tag{1.1}$$

where λ is a spectral parameter, α and β are arbitrary constants and

$$A=\begin{pmatrix} I_N & 0\\ 0 & -I_M\end{pmatrix},\qquad P(x,t)=\begin{pmatrix} 0 & Q(x,t)\\ R(x,t) & 0\end{pmatrix} \tag{1.2}$$

where I_N and I_M are identical square matrices of order N and M respectively, Q is an $N\times M$ rectangular matrix and R is an $M\times N$ rectangular matrix. Matrix elements of the potential $P(x, t)$ are elements of an infinite-dimensional commutative Z_2-graded algebra (superalgebra). We assume that $P(x, t) \to 0$ as $|x| \to \infty$.

We find the general form of nonlinear evolution equations integrable by the bundle (1.1) and construct the infinite-dimensional group of Bäcklund transformations for these equations. We show that all the equations integrable by (1.1) are Hamiltonian. We calculate the infinite family of Poisson brackets connected with these equations.

In the paper we consider the relativistic-invariant equations integrable by the bundle (1.1). Among these relativistic-invariant equations are both new ones and equations which are equivalent to already known equations. In particular, the integrable equation of the form (3.9) at $N = p = 2$, $M = q = 1$, $\Omega(L^+) = \frac{1}{4}(L^+)^{-1}$ and some special $P(x, t)$ is equivalent to the equations of the massive Thirring model with anticommuting fields.

We also discuss the structure of the Bäcklund transformations group. We introduce some special Bäcklund transformations—the so-called elementary Bäcklund transformations. An arbitrary discrete Bäcklund transformation is a product of the elementary Bäcklund transformations. We obtain the nonlinear superposition principle. It permits us to calculate the infinite family of the solutions of the integrable equations in a purely algebraic way. The gauge equivalence of the bundle (1.1) to the linear bundle and some other bundles is also discussed.

Let us emphasise that the nonlinear evolution equations integrable by (1.1) contain in the general case both classic boson fields and classic anticommuting fermion fields.

The paper is organised as follows. In § 2 we obtain some important relations and calculate the recursion operators. The general form of the integrable equations and their Bäcklund transformations is found in § 3. In § 4 the Hamiltonian structure of the integrable equations is discussed. The relativistic-invariant equations are considered in § 5. The equivalence of the bundle (1.1) to the linear bundle is demonstrated in § 6. In § 7 the structure of the Bäcklund transformations group is discussed and the elementary Bäcklund transformations are calculated. The nonlinear superposition principle is obtained in § 8. In the conclusion some reductions of the general bundle (1.1) are discussed.

2. Preliminary relations and recursion operator

2.1.

For convenience we present here some definitions and notations concerning graded algebras (see e.g. Berezin 1966, 1979, Kac 1977, Leites 1979). An algebra g is called a Z_2-graded algebra (superalgebra) if it admits a decomposition into a tensor sum $g = g_0 \oplus g_1$ of even (g_0) and odd (g_1) components. To any homogeneous $b \in g$ one assigns a number $\delta(b)$ (parity) which can take two values: 0 or 1. An element b is called even if $\delta(b) = 0$ and odd if $\delta(b) = 1$. The component g_0 consists of the even elements and the component g_1 contains the odd elements of g. A Z_2-graded algebra is called commutative if for any $a, b \in g$ the equality $[a, b] \overset{\text{def}}{=} ab - (-1)^{\delta(a)\delta(b)} ba = 0$ is satisfied.

We will assume that the elements of the potential $P(x, t)$ belong to the infinite-dimensional commutative Z_2-graded algebra, i.e.

$$P_{ik}(x, t) P_{ln}(x, t) = (-1)^{\delta(P_{ik})\delta(P_{ln})} P_{ln}(x, t) P_{ik}(x, t) \qquad (i, k, l, n = 1, \ldots, N+M).$$

So the even elements of $P(x, t)$ are classic boson fields and the odd elements are anticommuting variables (classic anticommuting fermion fields).

We will follow Berezin (1979), Kac (1977) and Leites (1979) and represent matrices of order $N+M$ with Z_2-graded algebra valued elements in the form $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α is a square $p \times p$ matrix, δ is a square $q \times q$ matrix, β is a rectangular $p \times q$ matrix, and γ is a rectangular $q \times p$ matrix $(q+p = N+M)$. The matrices α and δ consist of even elements of P. The matrices β and γ contain odd elements of P. A space of matrices of such type is denoted by $\mathrm{Mat}(p, q)$. The algebra $\mathrm{Mat}(p, q)$ is isomorphic to the superalgebra $\mathrm{gl}(p, q)$ (see e.g. Kac 1977, Leites 1979). The parity $\delta(P_{ik})$ of the element P_{ik} of the matrix P can be represented as follows: $\delta(P_{ik}) = \delta(i) + \delta(k) \pmod 2$, where $\delta(i) = 0$ for $1 \leq i \leq p$, $\delta(i) = 1$ for $p < i \leq M+N$. The matrices $P \in \mathrm{Mat}(p, q)$ have many properties which are analogous to the properties of the usual matrices (see e.g. Kac 1977, Leites 1979). In particular, the usual matrix trace has an analogue which is called the supertrace and defined as follows: $\mathrm{str}\, P \overset{\mathrm{def}}{=} \sum_{i=1}^{N+M} (-1)^{\delta(i)} P_{ii}$. For the supertrace we have $\mathrm{str}\,(PQ) = (-1)^{\delta(P)\delta(Q)} \mathrm{str}(QP)$. In the present paper we will deal only with even matrices $(\delta(P) = 0)$.

So we assume that potential $P(x, t) \in \mathrm{Mat}(p, q)$ $(p+q = N+M)$. In the general case rectangular matrices Q and R contain both even and odd elements. At $p = N$ and $q = M$ the matrices Q and R have only odd (anticommuting) variables.

Further, let ϕ be an arbitrary square matrix of order $N+M$. Let us represent it in the form $\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix}$, where ϕ_1 and ϕ_4 are respectively $N \times N$ and $M \times M$ matrices and ϕ_2 and ϕ_3 are respectively $N \times M$ and $M \times N$ rectangular matrices. We will denote $\phi_0 \overset{\mathrm{def}}{=} \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_4 \end{pmatrix}$ and $\phi_{\mathrm{F}} = \begin{pmatrix} 0 & \phi_2 \\ \phi_3 & 0 \end{pmatrix}$. Let us note that $(\phi_0 \psi_0)_{\mathrm{F}} = 0$, $(\phi_0 \psi_{\mathrm{F}})_{\mathrm{F}} = \phi_0 \psi_{\mathrm{F}}$ and $(\phi_{\mathrm{F}} \psi_{\mathrm{F}})_{\mathrm{F}} = 0$. These properties of the decomposition $\phi = \phi_0 + \phi_{\mathrm{F}}$ will be used often in what follows. The decomposition $\phi = \phi_0 + \phi_{\mathrm{F}}$ is the Fitting decomposition of the matrix superalgebra $\mathrm{Mat}(p, q)$ with respect to the matrix A (at $q = 0$ see e.g. Konopelchenko 1980a, b, 1981a). For the supermatrix of the potential $P(x, t) = P_{\mathrm{F}}(x, t)$.

Now we proceed to the construction of transformations and evolution equations connected with the bundle (1.1). At $q = 0$ the bundle (1.1) has been considered (briefly) by Konopelchenko (1981d) and the linear bundle with Z_2 grading was considered by Konopelchenko (1980c). Since the main steps of our calculations are the same as in Konopelchenko (1980a, b, c, 1981a, c, d) we will omit most of the intermediate calculations.

Let us introduce the fundamental matrix-solutions (assuming $P(x, t) \to 0$ as $|x| \to \infty$) $F^+(x, t, \lambda)$ and $F^-(x, t, \lambda)$ given by their asymptotic behaviour:

$$F^+(x, t, \lambda) \exp[-\mathrm{i}(\alpha\lambda^2 + 2\beta\lambda)x] \xrightarrow[x \to +\infty]{} 1,$$

$$F^-(x, t, \lambda) \exp[-\mathrm{i}(\alpha\lambda^2 + 2\beta\lambda)x] \xrightarrow[x \to -\infty]{} 1,$$

and the scattering matrix $S(\lambda, t)$:

$$F^+(x, t, \lambda) = F^-(x, t, \lambda) S(\lambda, t).$$

Matrices F^+, F^-, S have the same Z_2 structure, i.e. $F^+, F^-, S \in \mathrm{Mat}(p, q)$.

Let P and P' be two different potentials and F^+, $F^{+\prime}$ be the corresponding solutions of the problem (1.1). One can show that the following relation holds:

$$S'(\lambda, t) - S(\lambda, t) = -\mathrm{i}(\alpha\lambda + \beta) \int_{-\infty}^{+\infty} \mathrm{d}x\, (F^+(x, t, \lambda))^{-1} (P'(x, t) - P(x, t)) (F^+(x, t, \lambda))'. \tag{2.1}$$

By virtue of (1.1) there exists a correspondence between the transformations $P \to P'$ and transformations $S \to S'$. We will consider transformations of the scattering matrix only of the form

$$S(\lambda, t) \to S'(\lambda, t) = B^{-1}(\alpha\lambda^2 + 2\beta\lambda, t)S(\lambda, t)C(\alpha\lambda^2 + 2\beta\lambda, t) \tag{2.2}$$

where $B(\alpha\lambda^2+2\beta\lambda, t)$ and $C(\alpha\lambda^2+2\beta\lambda, t)$ are arbitrary supermatrices commuting with A, i.e. $B_0 = B$, $C_0 = C$.

Combining the relations (2.1) and (2.2) and taking into account the identity

$$\begin{aligned}(S^{-1}&(\lambda, t)(1 - B(\alpha\lambda^2 + 2\beta\lambda, t))S'(\lambda, t))_{\mathrm{F}} \\ &= -\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\partial}{\partial x}((F^+(x, t, \lambda))^{-1}(1 - B(\alpha\lambda^2 + 2\beta\lambda, t))(F^+(x, t, \lambda))')_{\mathrm{F}} \\ &= (\alpha\lambda + \beta) \int_{-\infty}^{+\infty} \mathrm{d}x((F^+(x, t, \lambda))^{-1}[P(x, t)(1 - B(\alpha\lambda^2 + 2\beta\lambda, t)) \\ &\quad - (1 - B(\alpha\lambda^2 + 2\beta\lambda, t))P'(x, t)](F^+(x, t, \lambda))')_{\mathrm{F}},\end{aligned}$$

we obtain

$$\begin{aligned}\int_{-\infty}^{+\infty} &\mathrm{d}x((F^+(x, t, \lambda))^{-1}(B(\alpha\lambda^2 + 2\beta\lambda, t)P'(x, t) \\ &- P(x, t)B(\alpha\lambda^2 + 2\beta\lambda, t))(F^+(x, t, \lambda))')_{\mathrm{F}} = 0.\end{aligned} \tag{2.3}$$

Let us represent the matrix $B(\alpha\lambda^2+2\beta\lambda, t)$ in the form $B(\alpha\lambda^2+2\beta\lambda, t) = \Sigma_{i=1}^{N^2+M^2} B_i(\alpha\lambda^2+2\beta\lambda, t)H_i$ where B_i are some functions, and the matrices H_i ($i = 1, \ldots, N^2+M^2$) form a basis of the subalgebra of matrices which commute with the matrix A. Rewriting the equality (2.3) over the components one obtains

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}x \,\mathrm{str}\Big(\sum_{i=1}^{N^2+M^2} (H_i P'(x, t) - P(x, t)H_i)B_i(\alpha\lambda^2 + 2\beta\lambda, t)\overset{++}{\tilde{\phi}}{}_{\mathrm{F}}^{(\mathrm{F})(in)}(x, t, \lambda)\Big) = 0, \\ \delta(i) = \delta(n)\end{aligned} \tag{2.4}$$

where

$$(\overset{++}{\tilde{\phi}}{}^{(in)}(x, t, \lambda))_{lk} = (F^+(x, t, \lambda))'_{ln}(F^+(x, t, \lambda))^{-1}_{ik} \qquad (i, k, l, n = 1, \ldots, N+M).$$

2.2 Recursion operator

Using equation (1.1) and the corresponding equation for F^{-1}, one can obtain the following equation for the quantity $\overset{++}{\tilde{\phi}}$:

$$\begin{aligned}\frac{\partial\overset{++}{\tilde{\phi}}(x, t, \lambda)}{\partial x} &= \mathrm{i}(\alpha\lambda^2 + 2\beta\lambda)[A, \overset{++}{\tilde{\phi}}(x, t, \lambda)] + \mathrm{i}(\alpha\lambda + \beta) \\ &\times (P'(x, t)\overset{++}{\tilde{\phi}}(x, t, \lambda) - \overset{++}{\tilde{\phi}}(x, t, \lambda)P(x, t)).\end{aligned} \tag{2.5}$$

Expressing the quantity ϕ_0 through the quantity ϕ_{F} and taking into account that $\overset{++}{\tilde{\phi}}{}_0^{\mathrm{F}}$ $(x = +\infty, t, \lambda) = 0$ and $[A, \phi_{\mathrm{F}}] = 2A\phi_{\mathrm{F}}$, we obtain ($\chi \overset{\mathrm{def}}{=} \overset{++}{\tilde{\phi}}{}_{\mathrm{F}}^{\mathrm{F}}$)

$$A\partial\chi/\partial x = 2\mathrm{i}(\alpha\lambda^2 + 2\beta\lambda)(1 - \mathrm{i}\alpha\mathscr{I})\chi + 2\beta^2\mathscr{I}\chi \tag{2.6}$$

where

$$\mathscr{J}\chi=\tfrac{1}{2}AP'(x)\int_x^\infty \mathrm{d}y\,(P'(y)\chi(y)-\chi(y)P(y))_0$$
$$-\tfrac{1}{2}A\int_x^\infty \mathrm{d}y\,(P'(y)\chi(y)-\chi(y)P(y))_0P(x).$$

As a result

$$\Lambda\chi(\lambda)=(\alpha\lambda^2+2\beta\lambda)\chi(x,t,\lambda) \tag{2.7}$$

where

$$\Lambda=(1-\mathrm{i}\alpha\mathscr{J})^{-1}\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2\mathscr{J}\right)=\sum_{l=0}^{\infty}(\mathrm{i}\alpha)^l\mathscr{J}^l\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2\mathscr{J}\right).$$

By virtue of (2.7), for any function $B_i(\alpha\lambda^2+2\beta\lambda,t)$ which is entire on $\alpha\lambda^2+2\beta\lambda$ one has

$$B_i(\alpha\lambda^2+2\beta\lambda,t)\chi(x,t,\lambda)=B_i(\Lambda,t)\chi(\lambda). \tag{2.8}$$

We will also need the operator Λ^+ adjoint to the operator Λ with respect to the bilinear form

$$\langle\phi,\psi\rangle=\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{str}(\phi_{\mathrm{F}}(x)\psi_{\mathrm{F}}(x)).$$

It is

$$\Lambda^+=(-\tfrac{1}{2}\mathrm{i}A\partial/\partial x+\mathrm{i}\beta^2\mathscr{J}^+)(1-\mathrm{i}\alpha\mathscr{J}^+)^{-1} \tag{2.9}$$
$$=\left(-\frac{\mathrm{i}}{2}A\frac{\partial}{\partial x}+\mathrm{i}\beta^2\mathscr{J}^+\right)\sum_{l=0}^{\infty}(\mathrm{i}\alpha)^l(\mathscr{J}^+)^l$$

where the operator $\mathscr{J}^+$ acts as follows:

$$\mathscr{J}^+\chi=-\tfrac{1}{2}P(x)\int_{-\infty}^{x}\mathrm{d}y\,(P(y)A\chi(y)-A\chi(y)P'(y))_0$$
$$+\tfrac{1}{2}\int_{-\infty}^{x}\mathrm{d}y\,(P(y)A\chi(y)-A\chi(y)P'(y))_0P'(x).$$

The operator Λ^+ plays a fundamental role in our further constructions. At $q=0$ the operators Λ and Λ^+ coincide with recursion operators given by Konopelchenko (1981d).

3. General form of the integrable equations and their Bäcklund transformations

Let us consider the relation (2.4) with entire functions $B_i(\alpha\lambda^2+2\beta\lambda)$. By virtue of (2.8) the equality (2.4) is equivalent to the following:

$$\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{str}\left(\sum_{i=1}^{N^2+M^2}(H_iP'(x,t)-P(x,t)H_i)B_i(\Lambda,t)\tilde{\phi}_{\mathrm{F}}^{++(\mathrm{F})(in)}\right)=0, \qquad \delta(i)=\delta(n). \tag{3.1}$$

From (3.1) one obtains

$$\int_{-\infty}^{+\infty} \mathrm{d}x \,\mathrm{str}\left[\overset{++}{\tilde{\phi}}{}_{\mathrm{F}}^{(\mathrm{F})(in)}(x, t, \lambda)\left(\sum_{i=1}^{N^2+M^2} B_i(\Lambda^+, t)(H_iP' - PH_i)\right)\right] = 0 \qquad (3.2)$$

where the operator Λ^+ is given by formula (2.9).

The equality (3.2) is fulfilled if

$$\sum_{i=1}^{N^2+M^2} B_i(\Lambda^+, t)(H_iP' - PH_i) = 0. \qquad (3.3)$$

Thus we find the transformations $P \to P'$ which correspond to the transformations $S \to S'$ of the scattering matrix of the form (2.2). These transformations are given by relation (3.3) where $B_i(\mu, t)$ are arbitrary functions entire on μ.

The transformations (2.2), (3.3) form an infinite-dimensional group. This group acts on the manifold of potentials $\{P(x, t)\}$ by formula (3.3) and on the manifold of the scattering matrices $\{S(\lambda, t)\}$ by formula (2.2). The group of transformations (2.2), (3.3) plays a fundamental role in the analysis of the nonlinear equations connected with the bundle (1.1) and their transformation properties.

The infinite-dimensional group of transformations (2.2), (3.3) contains transformations of various types. Let us consider a one-parameter subgroup of this group given by

$$B(\alpha\lambda^2 + 2\beta\lambda, t) = C(\alpha\lambda^2 + 2\beta\lambda, t) = \sum_{i=1}^{N^2+M^2} \exp\left(-\mathrm{i}\int_t^{t'} \mathrm{d}s\, \Omega_i(\alpha\lambda^2 + 2\beta\lambda, s)\right) H_i \qquad (3.4)$$

where $\Omega_i(\alpha\lambda^2 + 2\beta\lambda, t)$ are some (in general arbitrary) functions entire on $\alpha\lambda^2 + 2\beta\lambda$. It is not difficult to show that the transformation (2.2) with B and C given by (3.4) is a displacement in time t:

$$S(\lambda, t) \to S'(\lambda, t) = B^{-1}(\alpha\lambda^2 + 2\beta\lambda, t) S(\lambda, t) B(\alpha\lambda^2 + 2\beta\lambda, t) = S(\lambda, t'). \qquad (3.5)$$

The corresponding transformation of the potential is $P(x, t) \to P'(x, t) = P(x, t')$ and is determined by the relation†

$$\sum_{i=1}^{N^2+M^2} \exp\left(-\mathrm{i}\int_t^{t'} \mathrm{d}s\, \Omega_i(\Lambda^+, s)\right)(H_iP(x, t') - P(x, t)H_i) = 0 \qquad (3.6)$$

where in the operator Λ^+ one must put $P'(x, t) = P(x, s)$.

The relation (3.6) determines in implicit form the flow $Y_\Omega : P(x, t) \to P(x, t')$ or, in other words, the evolution system. This evolution system can also be described by a certain nonlinear evolution equation. Indeed, let us consider an infinitesimal displacement in time $t \to t' = t + \varepsilon$, $\varepsilon \to 0$. In this case $P(x, t') = P(x, t) + \varepsilon\, \partial P/\partial t$ and, keeping terms of the first order in ε, from (3.6) one obtains

$$\frac{\partial P(x, t)}{\partial t} - \mathrm{i} \sum_{i=1}^{N^2+M^2} \Omega_i(L^+, t)[H_i, P] = 0 \qquad (3.7)$$

where

$$L^+(P) \overset{\mathrm{def}}{=} \Lambda^+(P' = P),$$

† Transformations of such type were first considered by Calogero and Degasperis (1976) for the second-order linear bundle ($q = 0$).

i.e.

$$L^+ = (-\tfrac{1}{2}\mathrm{i}A\partial/\partial x + \mathrm{i}\beta^2 I^+)(1-\mathrm{i}\alpha I^+)^{-1}$$

where

$$I^+ = -\tfrac{1}{2}\left[P(x), \int_{-\infty}^{x} \mathrm{d}y\, [P(y), A\cdot]_0\right].$$

For the scattering matrix we correspondingly obtain

$$\mathrm{d}S(\lambda, t)/\mathrm{d}t = \mathrm{i}[Y(\lambda, t), S(\lambda, t)] \tag{3.8}$$

where

$$Y(\lambda, t) = \sum_{i=1}^{N^2+M^2} \Omega_i(\alpha\lambda^2 + 2\beta\lambda, t)H_i.$$

The nonlinear evolution equation (3.7) is an infinitesimal form of description of the flow Y_Ω. The relations (3.6) which do not contain the derivative $\partial P/\partial t$ are the 'integrated' form of the differential equations (3.7). The class of the evolution equations (3.7) is characterised by the integers N and M, by the recursion operator Λ^+ and by N^2+M^2 arbitrary entire functions $\Omega_1, \ldots, \Omega_{N^2+M^2}$.

The nonlinear evolution equations (3.7) are just the equations integrable by the IST method with the help of the bundle (1.1). If one generalises the IST method to the case of Z_2 grading, one can find the exact solutions of the equations (3.7).

One can show that the more general class of the integrable equations is connected with the bundle (1.1), namely the equations (3.7) with arbitrary functions $\Omega_i(\alpha\lambda^2+2\beta\lambda, t)$ meromorphic on $\alpha\lambda^2+2\beta\lambda$.

The class of the equations (3.7) contains a subclass of equations with $Y(\lambda, t) = \Omega(\alpha\lambda^2+2\beta\lambda, t)A$ where $\Omega(\alpha\lambda^2+2\beta\lambda, t)$ are arbitrary functions meromorphic on $\alpha\lambda^2+2\beta\lambda$. These equations are of the form

$$\partial P(x, t)/\partial t - 2\mathrm{i}\Omega(L^+, t)AP = 0. \tag{3.9}$$

In the particular case $\Omega(\mu, t) = -2\mu^2$ equation (3.9) is†

$$\mathrm{i}\frac{\partial P}{\partial t} + A\frac{\partial^2 P}{\partial x^2} + \left(2\beta^2 A - \mathrm{i}\alpha\frac{\partial}{\partial x}\right)P^3 = 0. \tag{3.10}$$

In terms of Q and R the equation (3.10) is a system

$$\mathrm{i}\frac{\partial Q}{\partial t} + \frac{\partial^2 Q}{\partial x^2} + \left(2\beta^2 - \mathrm{i}\alpha\frac{\partial}{\partial x}\right)QRQ = 0, \tag{3.11}$$

$$\mathrm{i}\frac{\partial R}{\partial t} - \frac{\partial^2 R}{\partial x^2} - \left(2\beta^2 + \mathrm{i}\alpha\frac{\partial}{\partial x}\right)RQR = 0,$$

where Q and R are respectively $N\times M$ and $M\times N$ rectangular supermatrices.

At $p=N$, $q=M$ matrices Q and R contain only anticommuting variables and the equations (39) describe pure fermionic classical systems. For example, at

$$M = q = 1, \quad N = p, \quad R = Q^+ = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}^+$$

† For calculations the relations $I^+AP = 0$, $(I^+)^2\partial P/\partial t = 0$ are useful.

and real α and β the system (3.11) is reduced to the N-component combined nonlinear Schrödinger equation with anticommuting fields:

$$\mathrm{i}\frac{\partial q_k}{\partial t}+\frac{\partial^2 q_k}{\partial x^2}+\left(2\beta^2-\mathrm{i}\alpha\frac{\partial}{\partial x}\right)q_k\sum_{n=1}^{N} q_n^+ q_n=0 \qquad (k=1,\ldots,N) \tag{3.12}$$

where

$$\begin{aligned} q_k(x,t)q_n(x,t)+q_n(x,t)q_k(x,t) &= q_k(x,t)q_n^+(x,t)+q_n^+(x,t)q_k(x,t) \\ &= q_k^+(x,t)q_n^+(x,t)+q_n^+(x,t)q_k^+(x,t)=0 \qquad (k,n=1,\ldots,N). \end{aligned}$$

For $\alpha=0$ equations of the form (3.11) with Z_2 grading have been considered by Kulish (1980b).

Let us consider now the transformations (3.3) with matrices B and C, commuting with the matrix $Y(\alpha\lambda^2+2\beta\lambda,t)$, i.e. $B=B_{\mathrm{O}(Y)}$, $C=C_{\mathrm{O}(Y)}$ and $\partial B//\partial t=\partial C/\partial t=0$. These transformations, as follows from (2.2), preserve the evolution law (3.8) of the scattering matrix. Therefore they transform the solutions of a certain equation of the form (3.7) into solutions of the same equation. Thus, the transformations (3.3) with $B=B_{\mathrm{O}(Y)}$ and $\partial B/\partial t=0$ are auto Bäcklund transformations for the equations (3.7). These auto Bäcklund transformations form an infinite-dimensional group.

The transformations (3.3) with $\partial B/\partial t\neq 0$ form an infinite-dimensional group of generalised Bäcklund transformations†; they transform the solutions of a certain equation of the form (3.7) into solutions of another equation of the form (3.7). In particular, the transformation (3.6) is the generalised Bäcklund transformation from the equation $\partial P/\partial t=0$ to equation (3.7).

Let us consider in conclusion some concrete equation of the form (3.7). We fix the matrix $Y(\lambda,t)$ and let $Y(\lambda,t)$ be a semisimple (i.e. diagonalisable) matrix. It is not difficult to show that due to the evolution law (3.8) the projection of the scattering matrix onto the subalgebra of matrices commuting with $Y(\lambda,t)$, is time independent:

$$\mathrm{d}S_{\mathrm{O}(Y)}/\mathrm{d}t=0.$$

Therefore at any λ the quantity $S_{\mathrm{O}(Y)}(\lambda)$ is an integral of motion. From this infinite set of inexplicit integrals of motion one can extract a countable set of explicit and local integrals of motion using the standard procedure (see e.g. Zakharov *et al* 1980, Konopelchenko 1981c, d).

4. Hamiltonian structure of the integrable equations

Let us consider the integrable equations of the form (3.9) with

$$\Omega(L^+,t)=\sum_{n=0}^{\infty}\omega_n(t)(L^+)^n \tag{4.1}$$

† Generalised Bäcklund transformations were first considered for the linear second-order bundle by Calogero and Degasperis (1976).

where $\omega_n(t)$ are arbitrary functions. The relation (2.1) plays an important role in the proof of the Hamiltonian character of equations (3.9). From (2.1) it follows that

$$\delta S_{nn}(\lambda) = -\mathrm{i}(-1)^{\delta(n)}(\alpha\lambda+\beta)\int_{-\infty}^{+\infty} \mathrm{d}x\ \mathrm{str}(\delta P(x,t)\overset{-+}{\phi}{}^{(nn)}(x,t,\lambda)) \qquad (n=1,\ldots,N+M) \tag{4.2}$$

where δP is an arbitrary variation of the potential $P(x,t)$, δS is the corresponding variation of the scattering matrix and $(\overset{-+}{\phi}{}^{(in)})_{kl} \overset{\mathrm{def}}{=} (F^+)_{kn}(F^-)^{-1}_{il}$. From (4.2) we obtain a basic variational equality

$$(\alpha\lambda+\beta)(\overset{-+}{\phi}{}^{(nn)}(x,t,\lambda))_{lk} = \mathrm{i}(-1)^{\delta(k)+\delta(n)}\frac{\vec{\delta}}{\delta P_{kl}(x,t)}S_{nn}(\lambda) \qquad (n,k,l=1,\ldots,N+M) \tag{4.3}$$

where $\vec{\delta}/\delta P$ denotes a left variational derivative (see Berezin 1966, 1979).

Further forming for the quantity $\overset{-+}{\phi}{}^{(in)}$ an equation analogous to the equation (2.5), we find that

$$\begin{aligned}(-\tfrac{1}{2}\mathrm{i}A\partial/\partial x+\mathrm{i}\beta^2I^+)A\overset{-+}{\phi}{}^{(in)}\\ &= (\alpha\lambda^2+2\beta\lambda)(1-\mathrm{i}\alpha I^+)A\overset{-+}{\phi}{}^{(in)}+\tfrac{1}{2}(\alpha\lambda+\beta)[P(x),\overset{-+}{\phi}{}_0^{(in)}(x=-\infty,t)]\end{aligned} \tag{4.4}$$

where

$$I^+\cdot = -\tfrac{1}{2}\left[P(x),\int_{-\infty}^{x}\mathrm{d}y[P(y),A\cdot]_0\right].$$

Since $(\overset{-+}{\phi}{}_0^{(nn)}(x=-\infty,t,\lambda))_{kl} = \delta_{kl}S_{nn}(\lambda)$ one obtains from (4.4) the relation

$$\begin{aligned}(-\tfrac{1}{2}\mathrm{i}A\partial/\partial x+\mathrm{i}\beta^2I^+)A\Pi(x,t,\lambda)\\ &= (\alpha\lambda^2+2\beta\lambda)(1-\mathrm{i}\alpha I^+)A\Pi(x,t,\lambda)-(\alpha\lambda+\beta)^2AP\end{aligned} \tag{4.5}$$

where

$$\Pi_{kl}(x,t,\lambda) = (\alpha\lambda+\beta)\sum_{n=1}^{N^2+M^2}A_{nn}\frac{(\overset{-+}{\phi}{}^{(nn)}(x,t,\lambda))_{kl}}{S_{nn}(\lambda)}. \tag{4.6}$$

Acting from the left on the equality (4.5) with the operator L^+, one obtains

$$[L^+-(\alpha\lambda^2+2\beta\lambda)]\tfrac{1}{2}\mathrm{i}\mathscr{D}\Pi = (\alpha\lambda+\beta)^2L^+AP \tag{4.7}$$

where

$$\mathscr{D}\cdot = \frac{\partial}{\partial x}+\beta^2\left[P(x),\int_{-\infty}^{x}\mathrm{d}y[P(y),\cdot\,]\right]. \tag{4.8}$$

Let us expand the left- and right-hand sides of the equality (4.7) into asymptotic series in $(\alpha\lambda^2+2\beta\lambda)^{-1}$. As a result we obtain the following system of relations:

$$\begin{aligned}&-\tfrac{1}{2}\mathrm{i}\mathscr{D}\Pi_{(0)} = \alpha L^+AP, \qquad \tfrac{1}{2}\mathrm{i}L^+\mathscr{D}\Pi_{(0)}-\tfrac{1}{2}\mathrm{i}\mathscr{D}\Pi_{(1)} = \beta^2L^+AP,\\ &L^+\mathscr{D}\Pi_{(k)} = \mathscr{D}\Pi_{(k+1)} \qquad (k=1,2,3,\ldots),\end{aligned} \tag{4.9}$$

where

$$\Pi(x, t, \lambda) = \sum_{n=0}^{\infty} (\alpha\lambda^2 + 2\beta\lambda)^{-n} \Pi_{(n)}(x, t).$$

From the recurrence relations (4.9) we have

$$(L^+)^n AP = \mathscr{D} \sum_{k=0}^{n-1} \left(-\frac{\mathrm{i}}{2\alpha}\right)\left(-\frac{\beta^2}{\alpha}\right)^{n-1-k} \Pi_{(k)} \qquad (n = 1, 2, 3, \ldots). \tag{4.10}$$

The relations (4.10) can be rewritten in the form

$$L^{+n} AP = L^{+q} \mathscr{D} \sum_{k=0}^{n-q-1} \left(-\frac{\mathrm{i}}{2\alpha}\right)\left(-\frac{\beta^2}{\alpha}\right)^{n-q-1-k} \frac{1}{(n-q)!} \frac{\partial^{n-q} \Pi(x, t, \lambda)}{\partial[(\alpha\lambda^2 + 2\beta\lambda)^{-1}]^{n-q}}\bigg|_{\lambda=\infty}$$

$$(n = 1, 2, 3, \ldots) \tag{4.11}$$

where q is an arbitrary integer.

With the use of (4.11) the evolution equation (3.9) with the function $\Omega(L^+)$ of the form (4.1) can be represented as

$$\partial P(x, t)/\partial t = (L^+)^q \mathscr{D} \sum_{n=0}^{\infty} \omega_n(t)$$

$$\times \sum_{k=0}^{n-q} \frac{1}{\alpha} \left(-\frac{\beta^2}{\alpha}\right)^{n-q-1-k} \frac{1}{(n-q)!} \frac{\partial^{n-q} \Pi(x, t, \lambda)}{\partial[(\alpha\lambda^2 + 2\beta\lambda)^{-1}]^{n-q}}\bigg|_{\lambda=\infty}. \tag{4.12}$$

Let us note now that for the quantity $\Pi(x, t, \lambda)$ from the relations (4.3) and (4.6) one has

$$\Pi(x, t, \lambda) = \mathrm{i}(\vec{\delta}/\delta P^{\mathrm{T}}(x, t)) \operatorname{str}[A \ln S_{\mathscr{D}}(\lambda)]\varepsilon \tag{4.13}$$

where $(S_{\mathscr{D}})_{ik} \overset{\text{def}}{=} \delta_{ik} S_{ii}$ and $\varepsilon_{kl} \overset{\text{def}}{=} \delta_{kl}(-1)^{\delta(k)}$. By virtue of (4.13) equation (4.12) is equivalent to the following:

$$\partial P/\partial t = (L^+)^q \mathscr{D} (\vec{\delta} \mathscr{H}_{-q}/\delta P^{\mathrm{T}})\varepsilon \tag{4.14}$$

where

$$\mathscr{H}_{-q} = \mathrm{i} \sum_{n=0}^{\infty} \omega_n(t) \sum_{k=0}^{n-q} \frac{1}{\alpha}\left(-\frac{\beta^2}{\alpha}\right)^{n-q-1-k} \frac{1}{(n-q)!} \frac{\partial^{n-q} \operatorname{str}[A \ln S_{\mathscr{D}}(\lambda)]}{\partial[(\alpha\lambda^2 + 2\beta\lambda)^{-1}]^{n-q}}\bigg|_{\lambda=\infty}. \tag{4.15}$$

Finally, it is not difficult to see that equation (4.14) is Hamiltonian; namely, it can be represented in the Hamiltonian form

$$\partial P(x, t)/\partial t = \{P(x, t), \mathscr{H}_{-q}\}_q \tag{4.16}$$

with respect to the infinite set of Hamiltonians $\mathscr{H}_{-q}$ (4.15) and Poisson brackets $\{\ \}_q$:

$$\{\mathscr{F}, \mathscr{H}\}_q \overset{\text{def}}{=} \int_{-\infty}^{+\infty} \mathrm{d}y \operatorname{str}\left(\mathscr{F} \frac{\overleftarrow{\delta}}{\delta P^{\mathrm{T}}(y, t)} (L^+)^q \mathscr{D} \frac{\vec{\delta} \mathscr{H}}{\delta P^{\mathrm{T}}(y, t)}\right) \tag{4.17}$$

where q is an arbitrary integer, $\mathscr{F}\overleftarrow{\delta}/\delta P$ denotes the right variational derivative of $\mathscr{F}$ (see e.g. Berezin 1966, 1979) and the operator $\mathscr{D}$ is given by formula (4.8).

The fact that the brackets (4.17) are indeed the Poisson brackets, i.e. for even functionals $\mathscr{F}$ and $\mathscr{H}$ they are skew-symmetric and satisfy the Jacobi identity, is proved by direct calculation. However, for odd functionals the brackets (4.17) are symmetric.

In the general case $\{\mathscr{F}, \mathscr{H}\}_q = (-1)^{\delta(\mathscr{F})\delta(\mathscr{H})}\{\mathscr{H}, \mathscr{F}\}_q$ and the brackets (4.17) satisfy a Z_2-graded version of the Jacobi identity (see e.g. Berezin (1979), Kac (1977)). Therefore (4.17) is a generalisation of the usual Poisson brackets to the case of the Z_2-graded algebra. The brackets (4.17) convert the algebra of functionals into a Lie superalgebra. In the mechanics of points such a generalisation of the Poisson brackets was considered by Berezin and Marinov (1977), Casalbuoni (1976), Berezin (1979).

In the particular case $\alpha = 0$, $\beta = 1$ the family of Poisson brackets (4.17) converts into the family of Poisson brackets connected with the Z_2-graded linear bundle with $P = \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$ (see Konopelchenko 1980c). At $q = 0$ the family of Poisson brackets (4.17) coincides with those given by Konopelchenko (1981d).

The fact that an infinite set of Poisson brackets is connected with integrable equations was first noted by Magri (1978; see also 1980). For the second-order linear bundle the hierarchy of Poisson brackets has been considered by Kulish and Reiman (1978). The hierarchies of Poisson brackets for integrable equations have been considered by Gelfand and Dorphman (1979, 1980), Reiman and Semenov-Tyan-Shansky (1980), Kulish (1980a), Konopelchenko (1981c, d, e, 1982).

Thus, it is shown that equations (3.9) are Hamiltonian with respect to the infinite set of Hamiltonian structures. The closed symplectic 2-forms which correspond to the Poisson brackets (4.17) are

$$\omega^{(q)}(\delta_1 P, \delta_2 P) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}(\delta_2 P \mathscr{D}^{-1}(L^+)^{-q}\delta_1 P - \delta_1 P \mathscr{D}^{-1}(L^+)^{-q}\delta_2 P)$$
$$q = 0, \pm 1, \pm 2, \ldots . \tag{4.18}$$

The two simplest Poisson brackets from the family (4.17) are

$$\begin{aligned}\{\mathscr{F}, \mathscr{H}\}_0 &= \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}\left(\frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}} \mathscr{D} \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}}\right) \\ &= \int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}\left(\frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} \frac{\partial}{\partial x} \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(x)} \right. \\ &\quad \left. + \beta^2 \frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} \left[P(x), \int_{-\infty}^{+\infty} \mathrm{d}y \left[P(y), \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(y)}\right]\right]\right),\end{aligned} \tag{4.19}$$

$$\begin{aligned}\{\mathscr{F}, \mathscr{H}\}_{-1} &= 2\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}\left(\frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} (1 - \alpha I^+) A \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(x)}\right) \\ &= 2\mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}\left(\frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} A \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(x)} \right. \\ &\quad \left. + \frac{\mathrm{i}\alpha}{2} \frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} \left[P(x), \int_{-\infty}^{x} \mathrm{d}y \left[P(y), \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(y)}\right]\right]\right).\end{aligned} \tag{4.20}$$

Let us introduce the brackets $\{\ ,\ \} \overset{\mathrm{def}}{=} \alpha\{\ ,\ \}_0 + \beta^2\{\ ,\ \}_{-1}$. It is easy to see that

$$\{\mathscr{F}, \mathscr{H}\} = \mathrm{i}\int_{-\infty}^{+\infty} \mathrm{d}x \, \mathrm{str}\left(\frac{\mathscr{F}\bar{\delta}}{\delta P^{\mathrm{T}}(x)} \left(2\beta^2 A - \mathrm{i}\alpha \frac{\partial}{\partial x}\right) \frac{\bar{\delta}\mathscr{H}}{\delta P^{\mathrm{T}}(x)}\right). \tag{4.21}$$

The brackets (4.21) are the Poisson brackets: the skew symmetry of $\{\mathscr{F}, \mathscr{H}\}$ is obvious, and the fulfilment of the Jacobi identity immediately follows from the independence of the kernel of the brackets (i.e. the operator $2\beta^2 A - i\alpha\partial/\partial x$) from $P(x, t)$.

5. Relativistic-invariant equations

In § 3 we considered examples of the equations (3.9) with entire functions $\Omega(L^+)$. For meromorphic functions $\Omega(L^+, t)$ the equations (3.9) can be rewritten in the equivalent form

$$g(L^+, t)\partial P/\partial t - 2if(L^+, t)AP = 0 \tag{5.1}$$

where $g(L^+, t)$ and $f(L^+, t)$ are entire functions on L^+ such that $\Omega(\mu, t) = f(\mu, t)/g(\mu, t)$.

Among the equations (3.9) with singular functions $\Omega(L^+, t)$, the equations (3.9) with $\beta = 0$ and $\Omega = \omega(L^+)^{-1}$ where ω is a constant are of the most interest. In this case, since $(L^+)^{-1}\cdot = 2i(1 - i\alpha I^+)\int_{-\infty}^{x} dy\,\cdot$, equation (3.9) is

$$\frac{\partial P(x,t)}{\partial t} + 4\omega\int_{-\infty}^{x} dy\,P(y,t) + 2i\omega\alpha\left[P(x), \int_{-\infty}^{x} dy\left[P(y), A\int_{-\infty}^{y} dz\,P(z)\right]\right] = 0. \tag{5.2}$$

Equation (5.2) is invariant under the Lorentz transformations

$$x \to x' = \rho x, \qquad t \to t' = \rho^{-1}t, \qquad P(x,t) \to P'(x',t') = \rho^{-1/2}P(x,t), \tag{5.3}$$

where ρ is a parameter of the Lorentz transformation.

Let us introduce a matrix $W(x, t)$ such that $P(x, t) = \partial W(x, t)/\partial x$. For the 'potential' $W(x, t)$ the equation (5.2) is a local one:

$$\frac{\partial^2 W(x,t)}{\partial x\,\partial t} + 4\omega W(x,t) - 2i\omega\alpha\left[\frac{\partial W(x,t)}{\partial x}, AW^2(x,t)\right] = 0. \tag{5.4}$$

We recall that $A = \begin{pmatrix} I_N & 0 \\ 0 & -I_M \end{pmatrix}$ and $W(x, t) \in \mathrm{Mat}(p, q)$. The equation (5.4) is Lorentz invariant too. Under the Lorentz transformations $W(x, t) \to W'(\rho x, \rho^{-1}t) = \rho^{1/2}W(x, t)$. In the components $W_1(x, t)$ and $W_2(x, t)$ $\left(W = \begin{pmatrix} 0 & W_1 \\ W_2 & 0 \end{pmatrix}\right)$ which are rectangular $N \times M$ and $M \times N$ matrices, equation (5.4) is the system

$$\begin{aligned} &\frac{\partial^2 W_1(x,t)}{\partial x\,\partial t} + 4\omega W_1 - 2i\omega\alpha\left(\frac{\partial W_1}{\partial x}W_2W_1 + W_1W_2\frac{\partial W_1}{\partial x}\right) = 0, \\ &\frac{\partial^2 W_2(x,t)}{\partial x\,\partial t} + 4\omega W_2 + 2i\omega\alpha\left(\frac{\partial W_2}{\partial x}W_1W_2 + W_2W_1\frac{\partial W_2}{\partial x}\right) = 0. \end{aligned} \tag{5.5}$$

For $M = N = 1$, $q = 0$ the system of equations (5.5) was first obtained by A V Mikhailov (see Gerdjikov *et al* 1980) and their connection with the quadratic bundle (1.1) ($N = M = 1$, $q = 0$) was discussed by Gerdjikov *et al* (1980).

System (5.5) contains as particular cases some interesting relativistic-invariant equations. For example, for real ω and α and under the reduction $W_1(x, t) = W_2^+(x, t) = U(x, t)$ the system (5.5) is equivalent to the matrix equation

$$\frac{\partial^2 U}{\partial x\,\partial t} + 4\omega U - 2i\omega\alpha\left(\frac{\partial U}{\partial x}U^+U + UU^+\frac{\partial U}{\partial x}\right) = 0. \tag{5.6}$$

For $M = 1$ equation (5.6) is ($U^T = (U_1, \ldots, U_N)$)

$$\frac{\partial U_i}{\partial x\,\partial t} + 4\omega U_i - 2i\omega\alpha\left(\frac{\partial U^i}{\partial x}\sum_{k=1}^{N} U_k^+U_k + U_i\sum_{k=1}^{N} U_k^+\frac{\partial U_k}{\partial x}\right) = 0 \qquad (i = 1, \ldots, N). \tag{5.7}$$

Let us note that among the fields $U_1, \ldots, U_N$ which satisfy the system of equations (5.7), p fields $U_1, \ldots, U_p$ are classic anticommuting fermion fields and $N-p$ fields $(U_{p+1}, \ldots, U_N)$ are classic boson fields. For $p=N$ the system of equations (5.7) describes the pure fermionic classic system. For $p=N+1$ equations (5.7) describe a pure bosonic system. In particular, for $N=1$, $p=2$ equations (5.7) reduce to the equation for one complex-valued boson field $U_1(x, t)$ with mass 4ω:

$$\frac{\partial^2 U_1(x,t)}{\partial x\, \partial t} + 4\omega U_1 - 2\mathrm{i}\omega\alpha U_1^* \frac{\partial}{\partial x}(U_1^2) = 0. \tag{5.8}$$

In the case $N=p=2$ and under the reduction $U_1(x,t)=\psi(x,t)$, $U_2(x,t)=\gamma\psi^+(x,t)$ where γ is an arbitrary constant, the system (5.7) reduces to the equation for one Grassmann-valued fermion field $\psi(x, t)$:

$$\frac{\partial^2 \psi(x,t)}{\partial x\, \partial t} + 4\omega\psi(x,t) - 2\mathrm{i}\omega\alpha|\gamma|^2\psi\psi^+ \frac{\partial\psi}{\partial x} = 0. \tag{5.9}$$

The field $\psi(x, t)$ satisfies the relations

$$\begin{aligned}\psi(x,t)\psi(y,t) + \psi(y,t)\psi(x,t) \\ &= \psi(x,t)\psi^+(y,t) + \psi^+(y,t)\psi(x,t) \\ &= \psi^+(x,t)\psi^+(y,t) + \psi^+(y,t)\psi^+(x,t) = 0.\end{aligned}$$

The relativistic-invariant equations (5.4)–(5.9) describe some two-dimensional (one time coordinate and one spatial coordinate) models of field theory, integrable by the IST method with the help of the bundle (1.1). These equations may have other applications too.

Let us now attract attention to the fact that equation (5.2) can be rewritten in the equivalent form (5.1), namely in the form

$$(1-\mathrm{i}\alpha I^+)^{-1}\frac{\partial P}{\partial t} = -\int_{-\infty}^{x} \mathrm{d}y\, P(y) \tag{5.10}$$

where we put $\omega = \frac{1}{4}$. The left-hand side of equation (5.10) is an infinite 'power' series in the operator I^+. This series $\Sigma_{l=0}^{\infty} (\mathrm{i}\alpha)^l (I^+)^l \partial P/\partial t$ contains only two non-vanishing terms (with $l=0, 1$) in the two cases.

First case: $N=M=1$, $q=0$, $\alpha=1$. In this case $I^{+2}=0$ and equation (5.10) at $P=\begin{pmatrix}0 & U\\ U^* & 0\end{pmatrix}$ is equivalent to the equations of the massive Thirring model (Gerdjikov *et al* 1980).

Second case: $N=2$, $M=1$, $p=2$, $\alpha=1$ and

$$P = \begin{pmatrix} 0 & 0 & \varphi \\ 0 & 0 & -\varphi^+ \\ \varphi^+ & -\varphi & 0 \end{pmatrix} \tag{5.11}$$

where $\varphi(x, t)$ is a Grassmann-valued variable and + denotes an involution in the Grassmann algebra (see e.g. Berezin 1966, 1979). In this case $I^{+2}\partial P/\partial t = 0$ and equation (5.10) reduces to the equation

$$\mathrm{i}\frac{\partial\varphi(x,t)}{\partial t} + \mathrm{i}\int_{-\infty}^{x} \mathrm{d}y\, \varphi(y,t) + \tfrac{1}{2}\int_{-\infty}^{x} \mathrm{d}y \frac{\partial}{\partial t}(\varphi(y,t)\varphi^+(y,t))\varphi(x,t) = 0 \tag{5.12}$$

plus the corresponding equation for $\varphi^+(x, t)$.

Let us introduce quantities $\psi_1(x,t)$ and $\psi_2(x,t)$:

$$\psi_1(x,t) \stackrel{\text{def}}{=} \tfrac{1}{2}\varphi(x,t)\exp\left(-\frac{i}{4}\int_{-\infty}^{x} dy\,\varphi(y,t)\varphi^+(y,t)\right),$$

$$\psi_2(x,t) \stackrel{\text{def}}{=} -\frac{i}{2}\exp\left(-\frac{i}{4}\int_{-\infty}^{x} dy\,\varphi(y,t)\varphi^+(y,t)\right)\int_{-\infty}^{x} dz\,\varphi(z,t). \tag{5.13}$$

From equation (5.12) it follows that

$$\int_{-\infty}^{x} dy\frac{\partial}{\partial t}(\varphi(y,t)\varphi^+(y,t)) = \int_{-\infty}^{x} dy\,\varphi^+(y,t)\int_{-\infty}^{y} dz\,\varphi(z,t) - \int_{-\infty}^{x} dy\,\varphi(y,t)\int_{-\infty}^{y} dz\,\varphi^+(z,t). \tag{5.14}$$

Further, the following identity holds:

$$\int_{-\infty}^{x} dy\,\varphi(y,t)\int_{-\infty}^{y} dz\,\varphi^+(z,t) - \int_{-\infty}^{x} dy\,\varphi^+(y,t)\int_{-\infty}^{y} dz\,\varphi(z,t) = \int_{-\infty}^{x} dy\,\varphi(y,t)\int_{-\infty}^{x} dz\,\varphi^+(z,t). \tag{5.15}$$

Let us recall that $\varphi(x,t)\varphi^+(y,t)+\varphi^+(y,t)\varphi(x,t)=0$. If we combine the relations (5.14), (5.15) and take into account the definition (5.13), we obtain

$$\int_{-\infty}^{x} dy\frac{\partial}{\partial t}(\varphi(y,t)\varphi^+(y,t)) = -4\psi_2(x,t)\psi_2^+(x,t). \tag{5.16}$$

With the use of (5.16) and the definitions (5.13) we obtain from (5.12) the following equations:

$$i\frac{\partial\psi_1(x,t)}{\partial t} - \psi_2 + \psi_1\psi_2^+\psi_2 = 0, \qquad i\frac{\partial\psi_2(x,t)}{\partial x} - \psi_1 + \psi_2\psi_1^+\psi_1 = 0. \tag{5.17}$$

Let us note that the first equation (5.17) follows from equation (5.12) and the second equation (5.17) is obtained directly from the definitions (5.13).

Equations (5.17) are just the equations of the massive Thirring model with anticommuting fields. The applicability of the IST method to these equations has been demonstrated by Izergin and Kulish (1978).

Thus, in the two cases considered equations (5.10) are equivalent to the equations of the massive Thirring model: in the first case with the fields $\psi_1(x,t)$, $\psi_2(x,t)$ which are the usual functions and in the second case with Grassmann-valued fields ψ_1, ψ_2.

Let us note that equation (5.9) with $\gamma=-1$, $\alpha=1$ and $\omega=\frac{1}{4}$ is equivalent to the equations of the massive Thirring model. Indeed, from equation (5.9), by introducing the quantity $\varphi(x,t)=\partial\psi(x,t)/\partial x$, we obtain equation (5.12). Then if we introduce the fields ψ_1 and ψ_2, by formulae (5.13) we obtain equation (5.17). Therefore equations (5.9) and (5.17) represent different forms of description of the same nonlinear system.

6. On the gauge equivalence of the bundle (1.1) to the linear bundle

The gauge equivalence of various equations integrable by the IST method is very useful for the analysis of these equations (see Zakharov and Mikhailov 1978, Zakharov and Takhtadjan 1979).

It was noted by A V Mikhailov (see Gerdjikov *et al* 1980) that the quadratic bundle (1.1) at $N = M = 1$, $q = 0$, $\beta = 0$ is gauge equivalent to the linear bundle. Here we present a generalisation of Mikhailov's result.

Let us consider the general bundle (1.1) with arbitrary N, M, p, q, α and β. Let us perform the gauge transformation

$$\psi(x, t, \lambda) \to \tilde{\psi}(x, t, \lambda) = V(x, t, \lambda)\psi(x, t, \lambda) \tag{6.1}$$

with $V = \begin{pmatrix} V_1 & 0 \\ V_2 & V_3 \end{pmatrix}$ where V_1 and V_3 are square matrices of order N and M respectively, V_2 is a rectangular $M \times N$ matrix, and they are equal to

$$\begin{aligned} V_1(x, t, \lambda) &= (\alpha\lambda + \beta)^{-1/2}\mathscr{P}_x\left\{\exp\left(\frac{\mathrm{i}\alpha}{2}\int_x^{-\infty} \mathrm{d}y\, Q(y, t)R(y, t)\right)\right\}, \\ V_2(x, t, \lambda) &= -\frac{\alpha}{2}(\alpha\lambda + \beta)^{-1/2}\mathscr{P}_x\left\{\exp\left(-\frac{\mathrm{i}\alpha}{2}\int_x^{-\infty} \mathrm{d}y\, R(y, t)Q(y, t)\right)\right\}R(x, t), \\ V_3(x, t, \lambda) &= (\alpha\lambda + \beta)^{1/2}\mathscr{P}_x\left\{\exp\left(-\frac{\mathrm{i}\alpha}{2}\int_x^{-\infty} \mathrm{d}y\, R(y, t)Q(y, t)\right)\right\}. \end{aligned} \tag{6.2}$$

Here $\mathscr{P}_x\{\exp \int^x \mathrm{d}y\, f(y)\}$ denotes a well known x-ordered exponent which is the solution of the matrix equation

$$\frac{\partial Z(x)}{\partial x} = f(x)Z(x) \qquad \left(Z(x) = \mathscr{P}_x\left\{\exp\int_{-\infty}^x \mathrm{d}y\, f(y)\right\}\right).$$

Let us note that

$$\left(\mathscr{P}_x\left\{\exp\int_{-\infty}^x \mathrm{d}y\, f(y)\right\}\right)^{-1} = \mathscr{P}_x\left\{\exp\int_x^{-\infty} \mathrm{d}y\, f(y)\right\}.$$

It is not difficult to verify that under the gauge transformation (6.1), (6.2) the bundle (1.1) converts into the linear bundle

$$\frac{\partial\tilde{\psi}}{\partial x} = \mathrm{i}\mu A\tilde{\psi} + \mathrm{i}\begin{pmatrix} 0 & \tilde{Q}(x, t) \\ \tilde{R}(x, t) & 0 \end{pmatrix}\tilde{\psi} \tag{6.3}$$

where

$$\mu = \alpha\lambda^2 + 2\beta\lambda, \qquad A = \begin{pmatrix} I_N & 0 \\ 0 & -I_M \end{pmatrix}$$

and $\tilde{Q}$, $\tilde{R}$ are rectangular $N \times M$ and $M \times N$ matrices which are equal to

$$\begin{aligned} \tilde{Q}(x, t) &= \mathscr{P}_x\left\{\exp\left(\frac{\mathrm{i}\alpha}{2}\int_x^{-\infty} \mathrm{d}y\, Q(y)R(y)\right)\right\}Q(x, t) \\ &\quad \times\mathscr{P}_x\left\{\exp\left(-\frac{\mathrm{i}\alpha}{2}\int_{-\infty}^x \mathrm{d}y\, R(y)Q(y)\right)\right\}, \\ \tilde{R}(x, t) &= \mathscr{P}_x\left\{\exp\left(-\frac{\mathrm{i}\alpha}{2}\int_x^{-\infty} \mathrm{d}y\, R(y)Q(y)\right)\right\} \\ &\quad \times(\beta^2 R(x, t) - \tfrac{1}{4}\alpha^2 R(x, t)Q(x, t)R(x, t) + \tfrac{1}{2}\mathrm{i}\alpha\,\partial R(x, t)/\partial x) \\ &\quad \times\mathscr{P}_x\left\{\exp\frac{\mathrm{i}\alpha}{2}\int_{-\infty}^x \mathrm{d}y\, QR\right\}. \end{aligned} \tag{6.4}$$

The gauge transformation (6.1), (6.2) is the generalisation of Mikhailov's gauge transformation to the case of the arbitrary quadratic bundle (1.1). At $\alpha=0$, $\beta=1$ the transformation (6.1), (6.2) is the identical one.

In two cases the x-ordered exponent converts into the usual exponent and the transformation (6.1) is simplified.

First case: $M=N=1$. At $q=0$ the matrices V_1, V_2 and V_3 are

$$V_1=\frac{1}{V_3}=(\alpha\lambda+\beta)^{-1/2}\exp\left(-\frac{i\alpha}{2}\int_{-\infty}^{x}dy\, Q(y,t)R(y,t)\right),$$
$$V_2=-\frac{\alpha}{2}(\alpha\lambda+\beta)^{-1/2}\exp\left(\frac{i\alpha}{2}\int_{-\infty}^{x}dy\, Q(y,t)R(y,t)\right)R(x,t) \tag{6.5}$$

and for the potential $\tilde{P}$ of the linear bundle (6.3) we have

$$\tilde{Q}(x,t)=Q(x,t)\exp\left(-i\alpha\int_{-\infty}^{x}dy\, Q(y,t)R(y,t)\right), \tag{6.6}$$
$$\tilde{R}(x,t)=\left(\beta^2R(x,t)-\frac{\alpha^2}{4}R^2(x,t)Q(x,t)+\frac{i\alpha}{2}\frac{\partial R(x,t)}{\partial x}\right)\exp\left(i\alpha\int_{-\infty}^{x}dy\, QR\right).$$

In the particular case $\beta=0$ we obtain Mikhailov's gauge transformation.

For $q=1$ the variables $Q(x,t)$ and $R(x,t)$ are Grassmann variables and

$$V_1=(\alpha\lambda+\beta)^{-1}V_3=(\alpha\lambda+\beta)^{-1/2}\exp\left(-\frac{i\alpha}{2}\int_{-\infty}^{x}dy\, Q(y,t)R(y,t)\right)$$
$$V_2=-\frac{\alpha}{2}(\alpha\lambda+\beta)^{-1/2}\exp\left(-\frac{i\alpha}{2}\int_{-\infty}^{x}dy\, Q(y,t)R(y,t)\right)R(x,t), \tag{6.7}$$
$$\tilde{Q}(x,t)=Q(x,t),\qquad \tilde{R}(x,t)=\tfrac{1}{2}(2\beta^2+i\alpha\partial/\partial x)R(x,t). \tag{6.8}$$

Second case: $N=2$, $M=1$, $p=2$ and

$$P=\begin{pmatrix}0&0&\varphi\\0&0&-\varphi^+\\\varphi^+&-\varphi&0\end{pmatrix} \tag{6.9}$$

where φ is a Grassmann variable. In this case due to $\varphi\varphi=\varphi^+\varphi^+=\varphi\varphi^++\varphi^+\varphi=0$ we have

$$QR=\varphi\varphi^+\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad RQ=0,$$

and therefore matrices V_1, V_2 and V_3 are equal to

$$V_1=(\alpha\lambda+\beta)^{-1/2}\exp\left(-\frac{i\alpha}{2}\sigma_3\int_{-\infty}^{x}dy\,\varphi(y,t)\varphi^+(y,t)\right),$$
$$V_2=-\frac{\alpha}{2}(\alpha\lambda+\beta)^{-1/2}(\varphi^+,-\varphi),\qquad V_3=(\alpha\lambda+\beta)^{1/2}, \tag{6.10}$$

where $\sigma_3 \overset{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The potential $\tilde{P}(x, t)$ of the linear bundle (6.3) is of the form

$$\tilde{P} = \begin{pmatrix} 0 & 0 & \tilde{\varphi}_1 \\ 0 & 0 & \tilde{\varphi}_2 \\ \tilde{\varphi}_3 & \tilde{\varphi}_4 & 0 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{\varphi}_1(x, t) &= \varphi(x, t) \exp\left(-\frac{i\alpha}{2} \int_{-\infty}^{x} dy\, \varphi(y, t)\varphi^+(y, t)\right), \\ \tilde{\varphi}_2(x, t) &= -\varphi^+(x, t) \exp\left(\frac{i\alpha}{2} \int_{-\infty}^{x} dy\, \varphi(y, t)\varphi^+(y, t)\right) = -\tilde{\varphi}_1^+, \\ \tilde{\varphi}_3(x, t) &= \tfrac{1}{2}\left(2\beta^2 + i\alpha \frac{\partial}{\partial x}\right)\varphi^+(x) \exp\left(\frac{i\alpha}{2} \int_{-\infty}^{x} dy\, \varphi(y, t)\varphi^+(y\ t)\right), \\ \tilde{\varphi}_4(x, t) &= -\tfrac{1}{2}\left(2\beta^2 + i\alpha \frac{\partial}{\partial x}\right)\varphi(x) \exp\left(-\frac{i\alpha}{2} \int_{-\infty}^{x} dy\, \varphi(y, t)\varphi^+(y, t)\right). \end{aligned} \tag{6.11}$$

Let us attract attention to the fact that at $\beta = 0$ the two cases considered above are just those two cases for which the relativistic-invariant equations (5.10) are equivalent to the equations of the massive Thirring model (see § 5).

Furthermore, in these two cases (first case: $N = M = 1$, $q = 0$, $\beta = 0$; second case: $N = 2$, $M = 1$, $p = 2$, $\beta = 0$ and potential P of the form (6.9)) the bundle (1.1) is gauge equivalent to the spectral problems which are used for integration of the massive Thirring model. The corresponding gauge transformations were given by Gerdjikov *et al* (1980) and Izergin and Kulish (1978).

7. The structure of the Bäcklund transformations group

In § 3 it was shown that the transformations (3.3) with $B = B_{O(Y)}$ and $\partial B/\partial t = 0$ form an infinite-dimensional group of auto Bäcklund transformations (BT) for the equations (3.7). The arbitrary entire functions $B_i(\Lambda^+)$ by which these BT are characterised can be represented in the form

$$B_i(\Lambda^+) = \prod_{k=0}^{n} (\Lambda^+ - \lambda_k^{(i)}) f_i(\Lambda^+) \tag{7.1}$$

where the functions $f_i(\Lambda^+)$ have no zeros and n is some integer. By virtue of (7.1) the arbitrary BT B is a combination of BT of two types:

$$B = B_d B_c$$

where B_d is a discrete BT, i.e. the BT (3.3) with functions $B_i(\Lambda^+) = \Pi_{k=1}^{n} (\Lambda^+ - \lambda_k^{(i)})$, and B_c is a continuous BT, i.e. the BT (3.3) with $B_i(\Lambda^+) = f_i(\Lambda^+)$.

Let us consider the discrete BT in more detail. Analogously to the case of the linear bundle (Konopelchenko 1979, 1981a, b), let us introduce the notion of the elementary BT (EBT). The EBT $B^{\alpha}_{\lambda_0^{(\alpha)}}$ is the BT (3.3) with functions B_i equal to ($r = \dim g_{O(Y)}$)

$$B_\alpha(\Lambda^+) = \Lambda^+ - \lambda_0^{(\alpha)}, \qquad B_1 = \ldots = B_{\alpha-1} = B_{\alpha+1} = \ldots = B_r \equiv 1. \tag{7.2}$$

An arbitrary discrete BT is a product of EBT:

$$B_d=\prod_{k=1}^{n_1} B^{(1)}_{\lambda^{(1)}_{0k}} \prod_{k=1}^{n_2} B^{(2)}_{\lambda^{(2)}_{0k}} \dots \prod_{k=1}^{n_r} B^{(r)}_{\lambda^{(r)}_{0k}}. \tag{7.3}$$

EBT are useful for investigation of the integrable equations (for the linear bundle see Konopelchenko (1981a, b)). Here we consider the simplest example $N=M=1$, $q=0$. In this case $H_1=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $H_2=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $P(x,t)=\begin{pmatrix}0&q\\r&0\end{pmatrix}$ and we have two EBT: $B^{(1)}_{\lambda^{(1)}_0}$ and $B^{(2)}_{\lambda^{(2)}_0}$. Let us consider first the EBT $B^{(1)}_{\lambda_0}$ $(B_1=\Lambda^+-\lambda_0, B_2\equiv 1)$. With the use of the explicit form of the operator Λ^+ and the relation

$$\mathscr{J}^+\begin{pmatrix}0&q'\\-r&0\end{pmatrix}=\frac{1}{2}\begin{pmatrix}0&q'\\-r&0\end{pmatrix}\int_{-\infty}^{x}\mathrm{d}y\,(rq-r'q'),$$

from (3.3) we obtain the following system of equations which define the transformation $B^{(1)}_{\lambda_0}$:

$$\begin{aligned}
&-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\left(q'\sum_{l=0}^{\infty}\left(\frac{\mathrm{i}\alpha}{2}\right)^{l}K_l(x)\right)+\frac{\mathrm{i}\beta^2}{2}q'\sum_{l=0}^{\infty}\left(\frac{\mathrm{i}\alpha}{2}\right)^{l}K_{l+1}(x)-\lambda_0 q'-q=0,\\
&-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\left(r\sum_{l=0}^{\infty}\left(\frac{\mathrm{i}\alpha}{2}\right)^{l}K_l(x)\right)-\frac{\mathrm{i}\beta^2}{2}r\sum_{l=0}^{\infty}\left(\frac{\mathrm{i}\alpha}{2}\right)^{l}K_{l+1}(x)+\lambda_0 r+r'=0,
\end{aligned} \tag{7.4}$$

where

$$K_l(x)\overset{\mathrm{def}}{=}\int_{-\infty}^{x}\mathrm{d}x_1(rq-r'q')\int_{-\infty}^{x_1}\mathrm{d}x_2\,(rq-r'q')\dots\int_{-\infty}^{x_{l-1}}\mathrm{d}x_l\,(rq-r'q').$$

Using the identity $K_l(x)=(1/l!)(K_1(x))^l$ $(l=1,2,3,\dots)$, one can rewrite the relations (7.4) in the form

$$-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\left[q'\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)\right]+\frac{\beta^2}{\alpha}q'\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)-\frac{\beta^2}{\alpha}q'-\lambda_0 q'-q=0, \tag{7.5a}$$

$$-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\left[r\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)\right]-\frac{\beta^2}{\alpha}r\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)+\frac{\beta^2}{\alpha}r+\lambda_0 r+r'=0. \tag{7.5b}$$

Integral terms in (7.5) can be transformed into local ones. Indeed, let us multiply the equality (7.5*a*) by $r\exp[\frac{1}{2}\mathrm{i}\alpha K_1(x)]$, the equality (7.5*b*) by $q'\exp[\frac{1}{2}\mathrm{i}\alpha K_1(x)]$, and then add the equalities obtained. As a result we have

$$-\frac{\mathrm{i}}{2}\frac{\partial}{\partial x}\left[q'r\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)\right]=(rq-r'q')\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)=-\frac{2\mathrm{i}}{\alpha}\frac{\partial}{\partial x}\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right). \tag{7.6}$$

From (7.6) one gets

$$\exp\left(\frac{\mathrm{i}\alpha}{2}K_1(x)\right)=\frac{2}{\alpha q'(x)r(x)}[1-(1-\alpha q'(x)r(x))^{1/2}]. \tag{7.7}$$

Substituting the equality (7.7) into (7.5), we finally obtain a system of relations which define the EBT $B^{(1)}_{\lambda_0}$:

$$\begin{aligned}
&\frac{1}{\alpha^2}\left(2\beta^2-\mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(\frac{1-(1-\alpha q'r)^{1/2}}{r}\right)-\left(\lambda_0+\frac{\beta^2}{\alpha}\right)q'-q=0,\\
&\frac{1}{\alpha^2}\left(2\beta^2+\mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(\frac{1-(1-\alpha q'r)^{1/2}}{q'}\right)-\left(\lambda_0+\frac{\beta^2}{\alpha}\right)r-r'=0.
\end{aligned} \tag{7.8}$$

In an analogous way one can calculate the EBT $B^{(2)}_{\lambda_0}$ ($B_1 \equiv 1$, $B_2 = \Lambda^+ - \lambda_0$). It is of the form

$$\frac{1}{\alpha^2}\left(2\beta^2 - \mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(\frac{1-(1-\alpha q r')^{1/2}}{r'}\right) - \left(\lambda_0 + \frac{\beta^2}{\alpha}\right)q - q' = 0,$$
$$\frac{1}{\alpha^2}\left(2\beta^2 + \mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(\frac{1-(1-\alpha q r')^{1/2}}{q}\right) - \left(\lambda_0 + \frac{\beta^2}{\alpha}\right)r' - r = 0. \qquad (7.9)$$

For $\alpha \to 0$ the EBT (7.8) and (7.9) are reduced to the corresponding EBT for the linear bundle ($\beta = 1$) (Konopelchenko 1979, 1981a, b):

$$B^{(1)}_{\lambda_0}(P \to P'): \quad \begin{aligned} &\frac{\partial q'}{\partial x} - \frac{1}{2\mathrm{i}} q'^2 r - 2\mathrm{i}\lambda_0 q' - 2\mathrm{i}q = 0, \\ &\frac{\partial r}{\partial x} + \frac{1}{2\mathrm{i}} r^2 q' + 2\mathrm{i}\lambda_0 r + 2\mathrm{i}r' = 0, \end{aligned}$$

and

$$B^{(2)}_{\lambda_0}(P \to P'): \quad \begin{aligned} &\frac{\partial q}{\partial x} - \frac{1}{2\mathrm{i}} q^2 r' - 2\mathrm{i}\lambda_0 q - 2\mathrm{i}q' = 0, \\ &\frac{\partial r'}{\partial x} + \frac{1}{2\mathrm{i}} r'^2 q + 2\mathrm{i}\lambda_0 r' + 2\mathrm{i}r = 0. \end{aligned}$$

The transformations (7.8) and (7.9) are the spatial parts of EBT and they are universal. The time parts of EBT are different for different equations (3.7) and their form can be found with the use of (7.8) or (7.9) and the explicit form of the equations.

It is not difficult to show that the EBT (7.8) and (7.9) commute—$B^{(1)}_{\lambda_0^{(1)}} B^{(2)}_{\lambda_0^{(2)}} = B^{(2)}_{\lambda_0^{(2)}} B^{(1)}_{\lambda_0^{(1)}}$. In particular $B^{(1)}_{\lambda_0} B^{(2)}_{\lambda_0} = B^{(2)}_{\lambda_0} B^{(1)}_{\lambda_0} = 1$. Therefore $B^{(2)}_{\lambda_0} = (B^{(1)}_{\lambda_0})^{-1}$ and $B^{(1)}_{\lambda_0} = (B^{(2)}_{\lambda_0})^{-1}$.

The simplest non-trivial non-elementary BT is

$$B_{\lambda_0^{(1)}, \lambda_0^{(2)}} \stackrel{\text{def}}{=} B^{(1)}_{\lambda_0^{(1)}} \cdot B^{(2)}_{\lambda_0^{(2)}}.$$

The explicit form of this BT can be found with the use of (7.8) and (7.9) by the formulae

$$B_{\lambda_0^{(1)}, \lambda_0^{(2)}}(P \to P'') = B^{(2)}_{\lambda_0^{(2)}}(P' \to P'')B^{(1)}_{\lambda_0^{(1)}}(P \to P') = B^{(1)}_{\lambda_0^{(1)}}(P' \to P'')B^{(2)}_{\lambda_0^{(2)}}(P \to P').$$

The transformation $B_{\lambda_0^{(1)}, \lambda_0^{(2)}}$ is given by the system of equations

$$\frac{1}{2\alpha}\left(2\beta^2 - \mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(q'K - \frac{q}{K}\right) - \left(\lambda_0^{(1)} + \frac{\beta^2}{\alpha}\right)q' + \left(\lambda_0^{(2)} + \frac{\beta^2}{\alpha}\right)q = 0,$$
$$\frac{1}{2\alpha}\left(2\beta^2 + \mathrm{i}\alpha\frac{\partial}{\partial x}\right)\left(rK - \frac{r'}{K}\right) - \left(\lambda_0^{(1)} + \frac{\beta^2}{\alpha}\right)r + \left(\lambda_0^{(2)} + \frac{\beta^2}{\alpha}\right)r' = 0, \qquad (7.10)$$

where $K(x) = \exp[\frac{1}{2}\mathrm{i}\alpha \int_{-\infty}^{x} \mathrm{d}y\, (rq - r'q')]$. This integral quantity K can be transformed into a local one. Namely, one can show that K is a solution of the algebraic equation

$$rq'K^4 + \frac{4}{\alpha}\left(\lambda_0^{(2)} + \frac{\beta^2}{\alpha}\right)K^3 - \left[r'q' + rq + \frac{4}{\alpha}\left(\lambda_0^{(1)} + \lambda_0^{(2)} + \frac{2\beta^2}{\alpha}\right)\right]K^2$$
$$+ \frac{4}{\alpha}\left(\lambda_0^{(1)} + \frac{\beta^2}{\alpha}\right)K + r'q = 0.$$

The explicit expression for K is cumbersome. We see that the simplest non-elementary BT is complicated enough and therefore the EBT are important.

8. Elementary Bäcklund transformations, nonlinear superposition principle and solutions of the integrable equations

The EBT is an effective tool for investigating the integrable equations. As we shall see, they allow us to construct an infinite family of solutions of equations (3.9) (at $N=M=1$, $q=0$).

Let us note first that the EBT (7.8) and (7.9) after certain transformations can be rewritten in the form

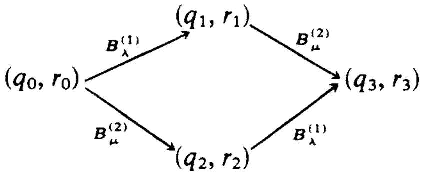
$$B_{\lambda_0}^{(1)}(P\to P'):\quad \begin{aligned} &\mathrm{i}\frac{\partial q'}{\partial x}+(\lambda_0 q'+q)(1+I)-\frac{\beta^2}{\alpha}(1-I)q'+\frac{\alpha q'}{2}(rq+r'q')=0,\\ &\mathrm{i}\frac{\partial r}{\partial x}-(\lambda_0 r+r')(1+I)+\frac{\beta^2}{\alpha}(1-I)r-\frac{\alpha r}{2}(rq+r'q')=0,\end{aligned} \tag{8.1}$$

where $I=(1-\alpha q'r)^{1/2}$ and

$$B_{\lambda_0}^{(2)}(P\to P'):\quad \begin{aligned} &\mathrm{i}\frac{\partial q}{\partial x}+(\lambda_0 q+q')(1+\tilde{I})-\frac{\beta^2}{\alpha}(1-\tilde{I})q+\frac{\alpha q}{2}(rq+r'q')=0,\\ &\mathrm{i}\frac{\partial r'}{\partial x}-(\lambda_0 r'+r)(1+\tilde{I})+\frac{\beta^2}{\alpha}(1-\tilde{I})r'-\frac{\alpha r'}{2}(rq+r'q')=0,\end{aligned} \tag{8.2}$$

where $\tilde{I}=(1-\alpha qr')^{1/2}$.

The commutativity of the EBT can be represented in the form of the following diagram:



(8.3)

where (q_0, r_0), (q_1, r_1), (q_2, r_2), (q_3, r_3) are the solutions of the definite equation of the form (3.9). The product $B_\mu^{(2)}B_\lambda^{(1)}$ of the EBT is given by the system of equations

$$B_{\lambda}^{(1)}(P_0\to P_1):\quad \begin{aligned} &\mathrm{i}\frac{\partial q_1}{\partial x}+(\lambda q_1+q_0)(1+I_{10})-\frac{\beta^2}{\alpha}(1-I_{10})+\frac{\alpha q_1}{2}(r_0q_0+r_1q_1)=0,\\ &\mathrm{i}\frac{\partial r_0}{\partial x}-(\lambda r_0+r_1)(1+I_{10})+\frac{\beta^2}{\alpha}(1-I_{10})-\frac{\alpha r_0}{2}(r_0q_0+r_1q_1)=0,\end{aligned} \tag{8.4}$$

$$B_{\mu}^{(2)}(P_1\to P_3):\quad \begin{aligned} &\mathrm{i}\frac{\partial q_1}{\partial x}+(\mu q_1+q_3)(1+I_{13})-\frac{\beta^2}{\alpha}(1-I_{13})q_1+\frac{\alpha q_1}{2}(r_1q_1+r_3q_3)=0,\\ &\mathrm{i}\frac{\partial r_3}{\partial x}-(\mu r_3+r_1)(1+I_{13})+\frac{\beta^2}{\alpha}(1-I_{13})r_3-\frac{\alpha r_3}{2}(r_1q_1+r_3q_3)=0,\end{aligned} \tag{8.5}$$

where $I_{il} \overset{\text{def}}{=} (1-\alpha q_i r_l)^{1/2}$. The transformation $B_\lambda^{(1)} B_\mu^{(2)}$ is given by the equations

$$B_\mu^{(2)}(P_0 \to P_2): \quad \begin{aligned} &\mathrm{i}\frac{\partial q_0}{\partial x} + (\mu q_0 + q_2)(1+I_{02}) - \frac{\beta^2}{\alpha}(1-I_{02})q_0 + \frac{\alpha q_0}{2}(r_0 q_0 + r_2 q_2) = 0, \\ &\mathrm{i}\frac{\partial r_2}{\partial x} - (\mu r_2 + r_0)(1+I_{02}) + \frac{\beta^2}{\alpha}(1-I_{02})r_2 - \frac{\alpha r_2}{2}(r_0 q_0 + r_2 q_2) = 0, \end{aligned} \tag{8.6}$$

$$B_\lambda^{(1)}(P_2 \to P_3) \quad \begin{aligned} &\mathrm{i}\frac{\partial q_3}{\partial x} + (\lambda q_3 + q_2)(1+I_{32}) - \frac{\beta^2}{\alpha}(1-I_{32})q_3 + \frac{\alpha q_3}{2}(r_2 q_2 + r_3 q_3) = 0, \\ &\mathrm{i}\frac{\partial r_2}{\partial x} - (\lambda r_2 + r_3)(1+I_{32}) + \frac{\beta^2}{\alpha}(1-I_{32})r_2 - \frac{\alpha r_2}{2}(r_2 q_2 + r_3 q_3) = 0. \end{aligned} \tag{8.7}$$

The equations (8.4)–(8.7) are a consequence of the commutativity of the EBT. In the previous section we have already used the systems of equations (8.4)–(8.7) for finding the soliton BT. Here we obtain another interesting consequence of equations (8.4)–(8.7).

If one compares the first equations (8.4) and (8.5) and the second equations (8.6) and (8.7), one obtains

$$\begin{aligned} &(\lambda q_1 + q_0)(1+I_{10}) - (\mu q_1 + q_3)(1+I_{13}) + \frac{\beta^2}{\alpha} q_1(I_{10} - I_{13}) + \frac{\alpha q_1}{2}(r_0 q_0 - r_3 q_3) = 0, \\ &(\lambda r_2 + r_3)(1+I_{32}) - (\mu r_2 + r_0)(1+I_{02}) + \frac{\beta^2}{\alpha} r_2(I_{32} - I_{02}) - \frac{\alpha r_2}{2}(r_0 q_0 - r_3 q_3) = 0. \end{aligned} \tag{8.8}$$

Equations (8.8) form an algebraic system and they give us the possibility, with given (q_0, r_0), (q_1, r_1), (q_2, r_2), to calculate (q_3, r_3) by purely algebraic operations. Thus the relations (8.8) are just the nonlinear superposition principle for equations (3.9): with three given solutions (q_0, r_0), (q_1, r_1) and (q_2, r_2), it allows us to calculate the fourth solution (q_3, r_3) in a purely algebraic way†.

The relations (8.8) give us the possibility to calculate by purely algebraic operations the infinite family of solutions of equations (3.9) for $M = N = 1$, $q = 0$. Indeed, let us start from the trivial solution $q = r = 0$ which we denote by P_0. Then let us act on this solution with all possible discrete BT (7.3) $(r = 2)$. As a result we obtain the infinite family of solutions

$$P_{(n_1,n_2)} = \begin{pmatrix} 0 & q_{(n_1,n_2)} \\ r_{(n_1,n_2)} & 0 \end{pmatrix}:$$

$$P_{(n_1,n_2)} \overset{\text{def}}{=} \prod_{i=1}^{n_1} B_{\lambda_i}^{(1)} \prod_{k=1}^{n_2} B_{\mu_k}^{(2)} P_0 \tag{8.9}$$

where n_1 and n_2 are arbitrary integers.

The solutions $P_{(n_1,0)}$ and $P_{(0,n_2)}$ are easily found directly from formulae (8.1) and (8.2). They are of the form

$$q_{(n_1,0)} = \sum_{k=1}^{n_1} \exp[2\mathrm{i}\Omega(\lambda_k)t + 2\mathrm{i}\lambda_k(x - x_{0k})], \qquad r_{(n_1,0)} = 0, \tag{8.10}$$

† Some nonlinear superposition formulae for certain integrable equations (e.g. sine-Gordon equation) are well known (see e.g. Miura 1976).

and

$$q_{(0,n_2)}=0, \qquad r_{(0,n_2)}=\sum_{k=1}^{n_2}\exp[-2\mathrm{i}\Omega(\mu_k)t-2\mathrm{i}\mu_k(x-\tilde{x}_{0k})] \tag{8.11}$$

where x_{0k} and $\tilde{x}_{0k}$ are arbitrary constants.

Using the solutions $P_{(n_1,0)}$ and $P_{(0,n_2)}$ we can calculate an arbitrary solution $P_{(n_1,n_2)}$ recursively with the help of the relations (8.8). Indeed, with given $P_{(0,0)}$, $P_{(1,0)}$ and $P_{(0,1)}$ we find $P_{(1,1)}$:

$$\begin{aligned} q_{(1,1)}=2(\lambda_1-\mu_1)\frac{(\alpha\mu_1+\beta^2)^{1/2}}{\alpha\lambda_1+\beta^2}&\exp[\mathrm{i}(\Omega(\lambda_1)+\Omega(\mu_1))t+\mathrm{i}(\lambda_1+\mu_1)x+\mathrm{i}\varphi_0]\\ &\times\frac{\cosh[\mathrm{i}(\Omega(\lambda_1)-\Omega(\mu_1))t+\mathrm{i}(\lambda_1-\mu_1)(x-x_{02})]}{\cosh^2[\mathrm{i}(\Omega(\lambda_1)-\Omega(\mu_1))t+\mathrm{i}(\lambda_1-\mu_1)(x-x_{01})]}, \end{aligned} \tag{8.12}$$

$$\begin{aligned} r_{(1,1)}=2(\mu_1-\lambda_1)\frac{(\alpha\lambda_1+\beta^2)^{1/2}}{\alpha\mu_1+\beta^2}&\exp[-\mathrm{i}(\Omega(\lambda_1)+\Omega(\mu_1))t-\mathrm{i}(\lambda_1+\mu_1)x-\mathrm{i}\varphi_0]\\ &\times\frac{\cosh[\mathrm{i}(\Omega(\lambda_1)-\Omega(\mu_1))t+\mathrm{i}(\lambda_1-\mu_1)(x-x_{01})]}{\cosh^2[\mathrm{i}(\Omega(\lambda_1)-\Omega(\mu_1))t+\mathrm{i}(\lambda_1-\mu_1)(x-x_{02})]}, \end{aligned} \tag{8.13}$$

where

$$x_{01}=-\frac{\mathrm{i}}{2}(\lambda_1-\mu_1)^{-1}\ln\frac{2a(\lambda_1-\mu_1)}{\lambda_1+\beta^2/\alpha},$$

$$x_{02}=-\frac{\mathrm{i}}{2}(\lambda_1-\mu_1)^{-1}\ln\frac{2a(\lambda_1-\mu_1)}{\mu_1+\beta^2/\alpha}$$

and a and φ_0 are arbitrary constants.

Further, from the solutions $P_{(0,1)}$, $P_{(0,2)}$ and $P_{(1,1)}$ with the use of formulae (8.8) we obtain $P_{(1,2)}$. From the solutions $P_{(0,1)}$, $P_{(1,1)}$, $P_{(2,0)}$ one obtains $P_{(2,1)}$. Then from the solutions $P_{(1,1)}$, $P_{(1,2)}$ and $P_{(2,1)}$ we find $P_{(2,2)}$. The continuation of this procedure gives us an arbitrary solution $P_{(n_1,n_2)}$.

Let us emphasise that with the use of the procedure described above we can find the solutions of equation (3.9) with an arbitrary function $\Omega(L^+)$.

Let us note that the solution $P_{(n_1,n_2)}$ is an algebraic function of the solutions $q_{(1,0)},\ldots,q_{(n_1,0)}$ and $r_{(0,1)},\ldots,r_{(0,n_2)}$ i.e. of the plane waves. In other words, the solutions $P_{(n_1,n_2)}$ of the nonlinear equations (3.9) are nonlinear superpositions of the solutions of the corresponding linearising equations.

The nonlinear superposition formulae (8.8) are simplified in the case $\alpha=0$, $\beta=1$. In this case equations (8.8) are reduced to

$$\begin{aligned} &2(\lambda-\mu)q_1+2(q_0-q_3)+\tfrac{1}{2}q_1^2(r_0-r_3)=0,\\ &2(\lambda-\mu)r_2+2(r_3-r_0)+\tfrac{1}{2}r_2^2(q_3-q_0)=0. \end{aligned} \tag{8.14}$$

From (8.14) one obtains

$$q_3=q_0+\frac{2(\lambda-\mu)}{r_2/2+2/q_1}, \qquad r_3=r_0-\frac{2(\lambda-\mu)}{q_1/2+2/r_2}. \tag{8.15}$$

Formulae (8.15) give the nonlinear superposition principle for equations (3.9) (for $\alpha=0$, $N=M=1$, $q=0$) integrable by the linear bundle (1.1) ($\alpha=0$, $p=1$, $N=M=1$, $q=0$).

9. Conclusion

In the present paper we consider the general structure and properties of the equations integrable by the bundle (1.1) in the general position, i.e. when the quantities Q and R are independent. However the possible reductions of the general equations are of great interest too. Let us point out two simple reductions.

First ($N = M$):

$$A = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}. \tag{9.1}$$

Second ($N = M$):

$$A = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & U \\ I_N & 0 \end{pmatrix} \tag{9.2}$$

where U is a square $N \times N$ matrix and I_N is the $N \times N$ identity matrix.

Under the reduction (9.1) the bundle (1.1) is equivalent to the bundle

$$\partial^2\chi/\partial x^2 + (\alpha\lambda + \beta)^2 Q^2 \chi - \mathrm{i}(\alpha\lambda + \beta)(\partial Q/\partial x)\chi + (\alpha\lambda^2 + 2\beta\lambda)^2\chi = 0 \tag{9.3}$$

where $\chi = \psi_1 + \psi_2$. Under the reduction (9.2) the bundle (1.1) is equivalent to

$$\partial^2\psi_2/\partial x^2 + (\alpha\tilde{\lambda} + \beta^2)U\psi_2 + \tilde{\lambda}^2\psi_2 = 0 \tag{9.4}$$

where $\tilde{\lambda} \overset{\text{def}}{=} \alpha\lambda^2 + 2\beta\lambda$.

The polynomial bundles (9.3) and (9.4) are generalisations of the well known spectral problems (with $\alpha = 0$). The general form of the integrable equations (3.9) under the reductions (9.1) and (9.2) and their Hamiltonian structures will be considered elsewhere.

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